

13.8: 8, 14, 18; 13 Review: 32, 34

8. $\operatorname{div} \mathbf{F} = 3x^2y - 2x^2y - x^2y = 0$, so $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 0 \, dV = 0$.

14. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + 1) \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 3(\rho^2 \sin^2 \phi + 1) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
 $= 2\pi \int_0^{\pi/2} \left[\frac{93}{5} \sin^3 \phi + 7 \sin \phi \right] d\phi = 2\pi \left[\frac{93}{5} (-\cos \phi + \frac{1}{3} \cos^3 \phi) - 7 \cos \phi \right]_0^{\pi/2} = \frac{194}{5} \pi$

18. As in the hint to Exercise 17, we create a closed surface $S_2 = S \cup S_1$, where S is the part of the paraboloid $x^2 + y^2 + z = 2$ that lies above the plane $z = 1$, and S_1 is the disk $x^2 + y^2 = 1$ on the plane $z = 1$ oriented downward, and we then apply the Divergence Theorem. Since the disk S_1 is oriented downward, its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and $\mathbf{F} \cdot (-\mathbf{k}) = -z = -1$ on S_1 . So $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} (-1) \, dS = -A(S_1) = -\pi$. Let E be the region bounded by S_2 . Then

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 1 \, dV = \int_0^1 \int_0^{2\pi} \int_1^{2-r^2} r \, dz \, d\theta \, dr = \int_0^1 \int_0^{2\pi} (r - r^3) \, d\theta \, dr = (2\pi) \frac{1}{4} = \frac{\pi}{2}.$$

Thus the flux of \mathbf{F} across S is $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}$.

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32. $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2) \, dV = \int_0^{2\pi} \int_0^1 \int_0^2 (3r^2 + 3z^2) r \, dz \, dr \, d\theta = 2\pi \int_0^1 (6r^3 + 8r) \, dr = 11\pi$

34. Here we must use Equation 13.8.6 since \mathbf{F} is not defined at the origin. Let S_1 be the sphere of radius 1 with center at the origin and outer unit normal \mathbf{n}_1 . Let S_2 be the surface of the ellipsoid with outer unit normal \mathbf{n}_2 and let E be the solid region between S_1 and S_2 . Then the outward flux of \mathbf{F} through the ellipsoid is given by

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS = -\iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1) \, dS + \iiint_E \operatorname{div} \mathbf{F} \, dV. \text{ But } \mathbf{F} = \mathbf{r}/|\mathbf{r}|^3, \text{ so}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot (|\mathbf{r}|^{-3} \mathbf{r}) = |\mathbf{r}|^{-3} (\nabla \cdot \mathbf{r}) + \mathbf{r} \cdot (\nabla |\mathbf{r}|^{-3}) = |\mathbf{r}|^{-3} (3) + \mathbf{r} \cdot (-3|\mathbf{r}|^{-4}) (\mathbf{r} |\mathbf{r}|^{-1}) = 0. \text{ (Here we have used}$$

Exercises 13.5.28(a) and 13.5.29(a).) And $\mathbf{F} \cdot \mathbf{n}_1 = \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = |\mathbf{r}|^{-2} = 1$ on S_1 . Thus

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS = \iint_{S_1} dS + \iiint_E 0 \, dV = (\text{surface area of the unit sphere}) = 4\pi(1)^2 = 4\pi.$$